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Geometry of Spaces of Polynomials

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Connections between the shape of the unit ball of a Banach space and analytic properties of the Banach space have been studied for many years. In this article, some geometric properties of spaces related to n-homogeneous polynomials are considered. In particular, the rotundity and smoothness of spaces of continuous n-homogeneous polynomials and its preduals are studied. Furthermore, an inequality relating the product of the norms of linear functionals on a Banach space with the norm of the continuous n-homogeneous polynomial determined by the product of the linear functionals is derived. This inequality is used to study the strongly exposed points of the predual of the space of continuous 2-homogeneous polynomials.

Relationships between geometric properties of Banach spaces and analytic properties of Banach spaces have been extensively studied for over 60 years [7, 8]. In the last 30 years, spaces of polynomials have become increasingly prominent, and significant efforts in infinite-dimensional holomorphy have led to many impressive results [1, 5, 10, 11, 15]. It is therefore tempting to consider the geometric structure of spaces of polynomials and their preduals. Some work has been done [1, 3, 4, 10], but the extreme point

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structure of the unit ball of these spaces and the differentiabilty of their norms seem to have avoided scrutiny. In this article, the notions of rotundity and smoothness of spaces of continuous n-homogeneous polynomials $\mathcal{P}(^nX)$ on a Banach space X and their preduals are considered. Also considered, and of interest for its own merits, is a relationship between the product of the norms of linear functionals on X and the norm of the product of the linear functionals considered as a continuous n-homogeneous polynomial. This inequality will prove useful in our study of extremal structure.

We recall that a *continuous n-homogeneous polynomial* on a real Banach space X is a function $P\colon X \to \mathbb{R}$ for which there exists a necessarily unique continuous, symmetric n-linear form $A\colon X^n \to \mathbb{R}$ with the property that $P(x) = A(x, \ldots, x)$ for every $x \in X$. We denote by $\mathcal{P}(^nX)$ the Banach space of all continuous n-homogeneous polynomials on X, where the norm is given by $\|P\| = \sup\{|P(x)| : \|x\| \leq 1\}$. The Banach space of continuous symmetric n-linear forms on X is denoted by $\mathcal{L}_s(^nX)$, the norm being given by $\|A\| = \sup\{|A(x_1, \ldots, x_n)| : \|x_i\| \leq 1\}$. The *polarization inequality*

$$\|P\| \leq \|A\| \leq \frac{n^n}{n!}\|P\|$$

shows that the spaces $\mathcal{P}(^nX)$ and $\mathcal{L}_s(^nX)$ are isomorphic. However, unless X is a Hilbert space, these spaces are not isometrically isomorphic for $n > 1$ [5, 11]. In fact, we shall see that their geometric structure can be quite different.

Recall that a normed linear space X is called *rotund* (or *strictly convex*) if the unit sphere of the space contains no nontrivial line segments; i.e., if $\|(x+y)/2\| < 1$ whenever $x, y \in X$, $x \neq y$, and $\|x\| = \|y\| = 1$. In other words, X is rotund if each element of the unit sphere of X is an extreme point of the unit ball of X. Under what conditions, if any, is $\mathcal{P}(^nX)$ rotund for $n \geq 2$? More generally, it would be nice to be able to characterize the extreme points of the unit ball of $\mathcal{P}(^nX)$.

First note that if z^* is a norm-one element in X^* that is not an extreme point of the unit ball of X^*, then the map $x \mapsto z^*(x)^n$, denoted by z^{*n}, is a norm-one element in $\mathcal{P}(^nX)$ that is not an extreme point of the unit ball of $\mathcal{P}(^nX)$. Indeed, if x^* and y^* are distinct norm-one elements in X^* and $z^* = (x^* + y^*)/2$ is also norm-one, then an application of the binomial theorem shows that z^{*n} is a nontrivial convex combination of norm-one elements in $\mathcal{P}(^nX)$. Thus z^{*n} is not an extreme point of the ball of $\mathcal{P}(^nX)$. This shows:

PROPOSITION 1. *If $\mathcal{P}(^nX)$ is rotund, then so is X^*.*

As a consequence of this fact, if X is any renorming of ℓ_∞, $\mathcal{P}(^nX)$ fails to be rotund since ℓ_∞ cannot be renormed to have a rotund dual [7]. In fact, even for spaces with rotund duals, it appears difficult for $\mathcal{P}({}^nX)$ to be rotund.

THEOREM 2. *If* $n \geq 2$ *and* $\dim X \geq 2$, *then* $\mathcal{P}({}^nX^*)$ *is not rotund*.

Proof. Let $\{b_1, b_2\} \subset X$ and $\{b_1^*, b_2^*\} \subset X^*$ satisfy $b_i^*(b_j) = \delta_{ij}$ for $i, j = 1, 2$. (Here δ_{ij} is the Kronecker delta.) Define $P(x^*) = x^*(b_1)^{n-1}x^*(b_2)$. Since P is weak* continuous and the unit ball of X^* is weak* compact, there exists x_0^* in X^* with $\|x_0^*\| = 1$ and $P(x_0^*) = \|P\|$. Choose x^{**} in X^{**} such that $x^{**}(x_0^*) = \|x^{**}\| = 1$ and define $Q(x^*) = x^{**}(x^*)^n$. Then Q and $P/\|P\|$ are both norm-one polynomials in $\mathcal{P}({}^nX^*)$ satisfying $Q(x_0^*) = (P/\|P\|)(x_0^*) = 1$. Since the zero set of Q is a subspace of X^* while the zero set of $P/\|P\|$ is not, $Q \neq P/\|P\|$. Then $\|\frac{1}{2}(Q + P/\|P\|)\| = \frac{1}{2}(Q(x_0^*) + P(x_0^*)/\|P\|) = 1$ implies that $\mathcal{P}({}^nX^*)$ is not rotund.

Although the above theorem does not guarantee that $\mathcal{P}({}^nX)$ always fails to be rotund, the ideas in the proof can be used to show that $\mathcal{P}({}^nX)$ fails to be rotund for all of the commonly considered spaces.

COROLLARY 3. *If* $n \geq 2$ *and* X *is a Banach space on which there is a norm-one projection onto a subspace* Y *of dimension at least two that is a dual space, then* $\mathcal{P}({}^nX)$ *is not rotund*.

Proof. Let $\pi : X \to Y$ be a norm-one projection from X onto Y. Then $P \mapsto P \circ \pi$ is an embedding of $\mathcal{P}({}^nY)$ into $\mathcal{P}({}^nX)$. Since rotundity is a hereditary property, it follows that $\mathcal{P}({}^nX)$ is not rotund.

We do not know if every infinite-dimensional Banach space admits a norm-one projection onto some subspace that is a dual with dimension greater than one. Since the adjoint map of such a projection is a linear Hahn–Banach extension operator from Y^* to X^*, the question is related to the following question: For each infinite-dimensional Banach space X, does there exist a subspace Y of X that is a dual space of dimension at least two and a linear Hahn–Banach extension operator from Y^* to X^*? If, additionally, Y is reflexive, then the existence of a linear Hahn–Banach extension operator T from Y^* to X^* implies the existence of a norm-one projection from X to Y. Indeed, $T^*|X$ is such a map [18]. The existence of one such reflexive subspace in every Banach space with rotund dual would be sufficient to show that $\mathcal{P}({}^nX)$ is never rotund. The above question appears to be an interesting question in its own right.

The above results ensure that the unit ball of $\mathcal{P}({}^nX)$ will contain nonextreme points in most cases. However, the results do not indicate what the extreme points of the unit ball of $\mathcal{P}({}^nX)$ look like. Giving a characterization of the extreme points of the unit ball of $\mathcal{P}({}^nX)$ is obviously a harder question. In [19], a characterization of the continuous 2-homogeneous polynomials on the n-dimensional Hilbert spaces ℓ_2^n is given, but the methods there do not appear to extend to the setting of non-Hilbert spaces. If one draws a picture of the unit ball of the three-dimensional space $\mathscr{P}(^2\ell_1^2)$, the continuous 2-homogeneous polynomials on two-dimensional ℓ_1, one finds that $\mathscr{P}(^2\ell_1^2)$ is the ℓ_1-sum of a two-dimensional space with a regular hexagon as its unit sphere and a two-dimensional space with a "bathtub" as its unit sphere. The connection between the extreme points of the unit ball of X^* and the extreme points of the unit ball of $\mathscr{P}(^2\ell_1^2)$ appears tenuous. A characterization of the extreme points of the unit ball of $\mathscr{P}(^2\ell_1^2)$ in terms of the eigenvalues of the symmetric 2×2-matrices representing the polynomials (as done in [19]) seems possible, but this is also far from clear.

Any clarity that arises from the above example stems not from looking at the space $\mathscr{P}(^2\ell_1^2)$, but from looking at its predual. The predual of $\mathscr{P}(^2\ell_1^2)$ is the space $\mathscr{P}_N(^2\ell_\infty^2)$ of nuclear polynomials on two-dimensional ℓ_∞. More will be said on the preduals of $\mathscr{P}(^nX)$ momentarily. For now, the only thing one needs to note is that the space $\mathscr{P}_N(^2\ell_\infty^2)$ turns out to be nothing more than the ℓ_∞-sum of a two-dimensional space with a regular hexagon as its unit sphere and a two-dimensional space with a "lens" as its unit sphere. The point is that, in this example, it seems clear from a picture where the extreme points come from: the extreme points are precisely the polynomials of the form $\pm x^2$, where x lies on the unit sphere of ℓ_1^2. That this is true more generally will be shown after a discussion of the predual.

In the example, it is clear that the space $\mathscr{P}(^2\ell_1^2)$ is a conjugate space since it is finite-dimensional. In fact, $\mathscr{P}(^nX)$ is always a conjugate space [11, 13, 17]. Let us briefly describe the duality.

The dual of the n-fold symmetric projective tensor product $\hat{\bigotimes}^n_{s,\pi}X$ is the Banach space $\mathscr{L}_s(^nX)$ of symmetric, continuous n-linear forms on X. Since this latter space is isomorphic to $\mathscr{P}(^nX)$, it follows that $\hat{\bigotimes}^n_{s,\pi}X$ can be considered as an "isomorphic" predual of $\mathscr{P}(^nX)$. The space $\hat{\bigotimes}^n_{s,\pi}X$ can be renormed so that it is an (isometric) predual, as follows: for $u \in \hat{\bigotimes}^n_{s,\pi}X$, define

$$\|u\|_\pi = \sup\{|A(u)| : A \in \mathscr{L}_s(^nX), \text{ and } \|\hat{A}\| = 1\},$$

where $\hat{A}$ denotes the n-homogeneous polynomial associated with A. Denote the tensor $x \otimes x \otimes \cdots \otimes x$ by x^n. Then, if u belongs to the uncompleted symmetric tensor product $\bigotimes^n_{s,\pi} X$, it follows from the polarization formula that u can be expressed as a linear combination of tensors of the form x^n. An alternative formula for $\|u\|_\pi$ can then be derived:

$$\|u\|_\pi = \inf\left\{\sum_{j=1}^m |\lambda_j|\|x_j\|^n : u = \sum_{j=1}^m \lambda_j x_j^n\right\}.$$

We denote the completion of $\bigotimes^n_{s,\pi} X$ with this norm by $\hat{X}_\pi^{(n)}$. Then $\mathscr{P}(^nX)$ is the dual space of $\hat{X}_\pi^{(n)}$. The mapping $x \mapsto x^n$ is a "universal" continuous

n-homogeneous polynomial on X: for every $P \in \mathcal{P}(^n X)$, there is a unique $\tilde{P} \in \hat{X}_\pi^{(n)*}$ with the same norm, such that

$$P(x) = \tilde{P}(x^n) \qquad \text{for every} \quad x \in X.$$

A continuous n-homogeneous P on X is said to be *nuclear* if there exist a bounded sequence $\{\varphi_j\}$ in X^* and $(\lambda_j) \in \ell_1$ such that $P(x) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(x)^n$ for every $x \in X$. The *nuclear norm* of P is given by

$$\|P\|_N = \inf\left\{\sum_{j=1}^{\infty} |\lambda_j| \, \|\varphi_j\|^n\right\},$$

the infimum being taken over all such representations of P. With this norm, the set of continuous nuclear n-homogeneous polynomials forms a Banach space which we denote by $\mathcal{P}_N(^n X)$. For spaces X whose dual has the approximation property, $\widehat{X^*}_\pi^{(n)}$ is isometrically isomorphic to the space $\mathcal{P}_N(^n X)$—we identify the tensor φ^n with the n-homogeneous polynomial $x \mapsto \varphi(x)^n$. We refer to [11] and [17] for further details.

We are now in a position to generalize the examples considered earlier. Recall that a point x_0 in the unit ball B_X of a normed linear space X is an *exposed point* of the unit ball if there exists a supporting hyperplane of the unit ball of X that touches the ball only at x_0; i.e., if there exists $x^* \in X^*$ such that $x^*(x_0) > \sup\{x^*(x) : x \in B_X \setminus \{x_0\}\}$. It is easy to see that every exposed point of the ball is also an extreme point of the ball.

Now, when X is a real normed space, the closed unit ball of $\hat{X}_\pi^{(n)}$ is the closed convex hull of the set $\{\pm x^n : x \in X, \|x\| = 1\}$. If X is finite-dimensional, this set is compact, and so it contains all of the extreme points of the unit ball $B_{\hat{X}_\pi^{(n)}}$. The next result shows that, for finite-dimensional spaces, the set of exposed points and the set of extreme points of the unit ball $B_{\hat{X}_\pi^{(n)}}$ coincide and equal this set.

THEOREM 4. *If $n \geq 2$ and X is a finite-dimensional, real normed linear space, then the set of exposed points of the unit ball of $\hat{X}_\pi^{(n)}$ is $\{\pm x^n : \|x\| = 1\}$.*

Proof. Fix a point $x_0 \in X$ with $\|x_0\| = 1$. We shall construct an n-homogeneous polynomial P on X such that the associated linear functional $\tilde{P}$ exposes the point x_0^n. We note first that if u is a norm-one element in $\hat{X}_\pi^{(n)}$ such that $u \neq \pm x^n$ for any x in the unit sphere of X, then u can be written as

$$u = \sum_{i=1}^{k} \lambda_i y_i^n + \lambda x_0^n, \tag{$*$}$$

where $\sum_{i=1}^{k} |\lambda_i| + |\lambda| = 1$, $\|x_0\| = \|y_i\| = 1$ and $y_i \neq \pm x_0$ for $i = 1, \ldots, k$, and $\lambda_i \neq 0$, $0 < |\lambda| < 1$. Choose $x^* \in S_{X^*}$ such that $x^*(x_0) = 1$, and let

Z denote the kernel of x^*. Define a projection $\pi : X \to Z$ by $\pi(x) = x - x^*(x)x_0$. Choose a polynomial $Q_2 \in \mathcal{P}(^2Z)$ satisfying

$$Q_2(z) \geq 0,$$

$$Q_2(z) = 0 \qquad \text{if and only if } z = 0,$$

and

$$Q_2(\pi(x)) < 1 \qquad \text{whenever } x \in X \text{ and } \|x\| \leq 1.$$

(For example, take an inner product norm, $\|.\|_2$, on Z such that $x \in X$ and $\|x\| \leq 1$ implies $\|\pi(x)\|_2 < 1$, and let $Q_2(x) = \|x\|_2{}^2$.)

Suppose first that $n = 2k$ is an even positive integer. Define $P \in \mathcal{P}(^nX)$ by $P(x) = x^*(x)^n - Q_2(\pi(x))^k$. Since

$$-1 < -Q_2(\pi(x))^k \leq P(x) \leq x^*(x)^n \leq 1 \qquad \text{for all } x \in B_X$$

and $P(x_0) = 1$, it follows that $\|P\| = 1$. Similarly, for $i = 1, \ldots, k$,

$$-1 < -Q_2(\pi(y_i))^k \leq P(y_i) = x^*(y_i)^n - Q_2(\pi(y_i))^k < x^*(y_i)^n \leq 1.$$

(The second of the strict inequalities in the above chain follows from $\pi(y_i) \neq 0$, since otherwise $y_i = \pm x_0$.)

Letting $\tilde{P}$ denote the linear functional on $\hat{X}_\pi^{(n)}$ determined by P, it follows that $\tilde{P}(x_0^n) = 1$, and, for all $u \in B_{\hat{X}_\pi^{(n)}} \setminus \{\pm x_0^n\}$,

$$|\tilde{P}(u)| \leq \sum_{i=1}^{k} |\lambda_i||P(y_i)| + |\lambda| < 1.$$

Therefore $\tilde{P}$ is a norm-one linear functional on $\hat{X}_\pi^{(n)}$ exposing x_0^n, and it follows that $\{\pm x^n : \|x\| = 1\}$ is a subset of the set of exposed points of the unit ball of $\hat{X}_\pi^{(n)}$. This combines with the remarks preceding the statement of the theorem to show that the sets actually coincide, and the result is proved in the case that n is even.

Now let $n = 2k + 1$ be an odd positive integer. Since $-x^n = (-x)^n$, it is clear that $B_{\hat{X}_\pi^{(n)}} = co\{\, x^n : \|x\| = 1\}$. Thus u can be written as in $(*)$, where, in addition to the restrictions there, $x^*(y_i) \geq 0$.

Define $P \in \mathcal{P}(^nX)$ by $P(x) = x^*(x)^n - Q_2(\pi(x))x^*(x)^{n-2}$, and again let $\tilde{P}$ be the linear functional on $\hat{X}_\pi^{(n)}$ determined by P. Then $P(x_0) = 1$, and, for $\|x\| \leq 1$,

$$|P(x)| \leq \max\left\{|x^*(x)|^n, Q_2(\pi(x))|x^*(x)|^{n-2}\right\} \leq 1,$$

since $x^*(x)^n$ and $Q(\pi(x))x^*(x)^{n-2}$ have the same sign. Thus $\|P\| = 1$.

If $x^*(y_i) = 0$ for some $i = 1, \ldots, k$, then $\tilde{P}(y_i^n) = 0$, and it is easy to check that $\tilde{P}(u) < 1$. So assume that u has the form in $(*)$ and $\sum_{i=1}^k |\lambda_i| + |\lambda| = 1$, $\lambda_i \neq 0$, $x^*(y_i) > 0$, $\|x_0\| = \|y_i\| = 1$, and $y_i \neq \pm x_0$ for $i = 1, \ldots, k$. As in the preceding case, for $i = 1, \ldots, k$,

$$\begin{aligned} -1 &< -Q_2(\pi(y_i)) \\ &\leq -Q_2(\pi(y_i))x^*(y_i)^{n-2} \\ &\leq P(y_i) \\ &= x^*(y_i)^n - Q_2(\pi(y_i))x^*(y_i)^{n-2} \\ &< x^*(y_i)^n \\ &\leq 1. \end{aligned}$$

Then

$$\tilde{P}(u) \leq \sum_{i=1}^k |\lambda_i||P(y_i)| + |\lambda| < \sum_{i=1}^k |\lambda_i| + |\lambda| = 1.$$

This shows, as in the preceding case, that $\tilde{P}$ exposes x_0^n and the set $\{\pm x^n : \|x\| = 1\}$ is contained in the set of exposed points of the unit ball of $\hat{X}_\pi^{(n)}$. This completes the proof of Theorem 4.

We have seen that, when X is finite-dimensional, the closed unit ball of $\hat{X}_\pi^{(n)}$ is the closed convex hull of the set of its extreme points, namely, $\{\pm x^n : \|x\| = 1\}$. Since $\hat{X}_\pi^{(n)}$ contains points that are not of the form z^n, $z \in X$, it follows that the unit sphere must have some points that are not extreme:

COROLLARY 5. *If X is a real, finite-dimensional normed space of dimension at least two, then $\hat{X}_\pi^{(n)}$ is not rotund.*

Another geometric property of a Banach space is smoothness. A normed linear space is *smooth* if there is a unique supporting hyperplane at every point on the unit sphere. For reflexive spaces, rotundity and smoothness are dual notions. In general, if X^* is smooth (rotund), then X is rotund (smooth), but the converses need not hold [7]. The next result follows immediately from Theorem 2 and the reflexivity of finite-dimensional spaces.

COROLLARY 6. *If $n \geq 2$ and X is finite-dimensional normed space of dimension at least two, then $\hat{X}_\pi^{(n)}$ is not smooth.*

Corollary 6 will also hold when X is an infinite-dimensional dual space with $\mathcal{P}(^nX)$ reflexive. Although such spaces are not common, some do exist [1].

In the same vein, Corollary 5 implies:

COROLLARY 7. *If $n \geq 2$ and X is a finite-dimensional, real normed space of dimension at least two, then $\mathcal{P}(^n X)$ is not smooth.*

In our analysis of the extreme points of the unit ball of $\hat{X}_\pi^{(n)}$, it was important to know that the set $\{x^n : \|x\| = 1\}$ was closed in $\hat{X}_\pi^{(n)}$. The next sequence of results considers what occurs if X is not finite-dimensional. The first goal in gaining some insight into this problem is an inequality relating the product of the norms of linear functionals on X and the norm of the product of the linear functionals in $\mathcal{P}(^n X)$. This will be given in Theorem 9 and is of interest for its own merits.

We begin with a geometric result. If x and y are two points on the unit sphere of an inner product space, it follows easily from the parallelogram identity that either $(x+y)/2$ or $(x-y)/2$ has a norm of at least $1/\sqrt{2}$. Thus, for a suitable choice of signs, every point in the convex hull of $\{\pm x,\ \pm y\}$ has a norm of at least $1/\sqrt{2}$. The result below generalizes this to any finite set of points on the unit sphere of any normed space. In particular, it is shown that for any n points $x_1, \dots, x_n$ on the unit sphere of a normed space, there exists a choice of signs for which the convex hull of $\pm x_1, \dots, \pm x_n$ is at least some fixed distance K_n from the origin. Furthermore, the constant K_n is independent of the ambient space.

THEOREM 8. *For each natural number n, there exists $K_n > 0$ with the following property: If X is a normed space and $x_1, \dots, x_n$ are on the unit sphere of X, then there exist $\varepsilon_1, \dots, \varepsilon_n$, where $\varepsilon_j = 1$ or -1 for $j = 1, \dots, n$, such that every point in the convex hull of the set $\{\varepsilon_1 x_1, \dots, \varepsilon_n x_n\}$ has a norm of at least K_n.*

Proof. Without loss of generality, assume that X has dimension n. We first prove a stronger statement in the context of inner product spaces: If H is an n-dimensional inner product space and $0 < \alpha \leq \beta$, then there exists $\delta_n > 0$ such that if $\alpha \leq \|x_j\| \leq \beta$ for $1 \leq j \leq n$, there exists a choice of signs $\varepsilon_j = \pm 1$ so that every point in the convex hull of $\{\varepsilon_1 x_1, \dots, \varepsilon_n x_n\}$ has a norm of at least δ_n. Indeed, suppose on the contrary that this fails to hold. Then, for every $k \in \mathbb{N}$ and every choice of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, there exists a sequence $(x_j^{(k)})_{j=1}^n$ with $\alpha \leq \|x_j^{(k)}\| \leq \beta$ and a point $x_\varepsilon^{(k)}$ in the convex hull of $\{\varepsilon_1 x_1^{(k)}, \dots, \varepsilon_n x_n^{(k)}\}$ with a norm of less than $1/k$. By compactness, we can assume that each of the sequences $(x_j^{(k)})$ converges to some x_j. Now, for each ε, $x_\varepsilon^{(k)} \to 0$, and thus the convex hull of $\varepsilon_1 x_1, \dots, \varepsilon_n x_n$ contains the origin for every ε. But this is impossible: Let H be a closed hyperplane that does not contain any x_j, and choose $\varepsilon_j = \pm 1$ so that all of the points $\varepsilon_j x_j$ are in the same open half-space; for this ε, $\mathrm{co}\{\varepsilon_1 x_1, \dots, \varepsilon_n x_n\}$ clearly does not contain the origin. Thus we have a contradiction.

For the general case, let X be an n-dimensional normed space. It is well known (see, for example, [14, p. 58]) that there exists an inner product norm, $|||\cdot|||$, on X such that

$$\|x\| \le |||x||| \le \sqrt{n}\ \|x\| \qquad \text{for every } x \in X.$$

Now let $x_1, \ldots, x_n$ lie in the unit sphere of X. Then $1 \le |||x_j||| \le \sqrt{n}$ for every j, and hence, by the first part of the proof, there exist $\varepsilon_j = \pm 1$ such that every $x \in \mathrm{co}\{\varepsilon_1 x_1, \ldots, \varepsilon_n x_n\}$ satisfies $|||x||| \ge \delta_n$, where δ_n is a positive constant depending only on n. Therefore,

$$\|x\| \ge \frac{\delta_n}{\sqrt{n}} \qquad \text{for every } x \in \mathrm{co}\{\varepsilon_1 x_1, \ldots, \varepsilon_n x_n\}.$$

This concludes the proof of Theorem 8.

Let δ_n be the largest number with the property that, given any n points $x_1, \ldots, x_n$ in an inner product space with $1 \le \|x_j\| \le \sqrt{n}$, there exist $\varepsilon_j = \pm 1$ such that every point in the convex hull of $\{\varepsilon_1 x_1, \ldots, \varepsilon_n x_n\}$ has a norm of at least δ_n. Clearly, the points x_j may be taken on the unit sphere in the definition of δ_n. Unfortunately, the proof gives no information about the size of these constants. In the case $n = 2$, it follows from the parallelogram identity that $\delta_2 \ge 1/\sqrt{2}$, and taking x_1 orthogonal to x_2 yields $\delta_2 = 1/\sqrt{2}$. Similarly, the consideration of n orthonormal vectors shows that $\delta_n \le 1/\sqrt{n}$. However, this is by no means the "worst" case. If we place $x_1, \ldots, x_n$ at n adjacent vertices of a regular $2n$-sided plane polygon, we find that $\delta_n \le \sin(\pi/2n)$. We are grateful to Pádraig Kirwan for pointing out that this is the "worst" case among all of the configurations in which the points all lie in a plane. We do not know a better estimate for δ_n.

If κ_n denotes the largest value of K_n for which Theorem 8 holds, the proof shows that $\kappa_n \ge \delta_n/\sqrt{n}$. For the case $n = 2$, it is easy to determine the value of κ_n. The estimate above gives $\kappa_2 \ge 1/2$, and with $X = \ell_\infty$ and $x_1 = e_1$ and $x_2 = e_2$, the first two canonical unit vectors, it is easy to see that $\kappa_2 \le 1/2$. Thus $\kappa_2 = \frac{1}{2}$.

Consider now the problem of putting a lower bound on the norm of a product of linear functionals. This will be done by relating this question to the geometric result given in Theorem 8. So as not to blind the reader by a plethora of stars, we change our notation for the next theorem only. If $\varphi_1, \ldots, \varphi_n$ are bounded linear functionals on a normed space X, then $\varphi_1\varphi_2\cdots\varphi_n$ denotes the continuous n-homogeneous polynomial from X into the scalars given by $x \mapsto \varphi_1(x)\varphi_2(x)\ldots\varphi_n(x)$.

THEOREM 9. *For each $n \in \mathbb{N}$, there exists $C_n > 0$ with the following property: If X is a normed space and $\varphi_1, \ldots, \varphi_n$ are bounded linear functionals on X, then*

$$C_n \|\varphi_1\|\|\varphi_2\|\cdots\|\varphi_n\| \le \|\varphi_1\varphi_2\cdots\varphi_n\|.$$

Proof. Without loss of generality, assume that $\|\varphi_1\| = \cdots = \|\varphi_n\| = 1$. Theorem 8 provides the existence of a choice of signs $\varepsilon_j = \pm 1$ such that the convex hull of $\{\varepsilon_1\varphi_1, \ldots, \varepsilon_n\varphi_n\}$ is disjoint from the open ball about the origin of radius K_n. Since the convex hull of a finite set is closed, the geometric form of the Hahn–Banach theorem provides a norm-one element x^{**} in X^{**}, such that $x^{**}(\varphi) \geq K_n$ for each $\varphi \in \mathrm{co}\{\varepsilon_1\varphi_1, \ldots, \varepsilon_n\varphi_n\}$. Goldstine's theorem then yields a norm-one $x \in X$ such that $\varphi(x) \geq K_n$ for every $\varphi \in \mathrm{co}\{\varepsilon_1\varphi_1, \ldots, \varepsilon_n\varphi_n\}$. Therefore, $\|\varphi_1\varphi_2\cdots\varphi_n\| \geq |\varphi_1\varphi_2\cdots\varphi_n(x)| \geq K_n^n$, which proves the theorem.

If γ_n denotes the largest constant with the property given in the statement of this theorem, the proof shows that

$$\gamma_n \geq (\kappa_n)^n \geq \left(\frac{\delta_n}{\sqrt{n}}\right)^n \qquad \text{for every } n.$$

This gives $\gamma_2 \geq 1/4$, and, taking $X = \ell_1$ with $\varphi_1 = e_1^*$ and $\varphi_2 = e_2^*$, we see that, in fact,

$$\gamma_2 = \tfrac{1}{4}.$$

Similarly, by considering the canonical functionals $e_1^*, \ldots, e_n^*$ on ℓ_1, we see that

$$\gamma_n \leq \frac{1}{n^n} \qquad \text{for } n > 2.$$

It is possible to define these constants for a specific normed space X as

$$\gamma_n(X) = \sup\big\{C > 0 : C\|\varphi_1\|\cdots\|\varphi_n\| \leq \|\varphi_1\varphi_2\cdots\varphi_n\| \\ \text{for all } \varphi_1, \ldots, \varphi_n \in X^*\big\}.$$

Benitez, Sarantopoulos, and Tonge [6] prove a more general result than Theorem 9, using different methods. They obtain better estimates for these constants in certain cases. They show that if X is a *complex* Banach space, then $\gamma_n(X) \geq 1/n^n$; while, if X is a complex Hilbert space and n a power of 2, then $1/n^n$ can be replaced by $1/n!$. Using a complexification argument, they show that, for a *real* inner product space, H,

$$\gamma_n \geq \frac{n!}{n^{2n}2^{(n-1)/2}} \qquad \text{for every } n.$$

The authors are indebted to the referee for pointing out the work of J. Arias-de-Reyna [2]. The referee mentions that it is shown in [2] that $\gamma_n(H) \geq n^{-n/2}$ if H is a complex Hilbert space. The referee also notes that it follows by complexification that $\gamma_n(H) \geq 2(2n)^{-n/2}$ if H is a real Hilbert space.

It is a trivial consequence of the Hahn–Banach theorem that, given any two points x, y in a normed space X, there is a continuous bilinear form A on X such that $\|A\| = 1$ and $A(x, y) = \|x\| \, \|y\|$. In terms of tensor products, we have $\|x \otimes y\| = \|x\| \, \|y\|$ in the projective tensor product $X \hat{\bigotimes}_\pi Y$. However, if the bilinear form A must also be *symmetric*, then the situation is quite different.

For example, consider the two-dimensional ℓ_1-space, ℓ_1^2. Every symmetric bilinear form is given by a symmetric matrix:

$$A(x, y) = (x_1 \quad x_2) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

It is easy to see that $\|A\| = \max\{|a|, \ |b|, \ |c|\}$. Letting $x = (1/2, \ 1/2)$ and $y = (1/2, \ -1/2)$ gives two norm-one elements in ℓ_1^2. But then, for any A with $\|A\| = 1$, $|A(x, y)| = |a - b|/4 \leq 1/2$.

On the other hand, if X is a real inner product space and x, y are on the unit sphere of X, we can choose a self-adjoint isometry $R : X \to X$ such that $Ry = x$, and then the symmetric bilinear form $A(x, y) = \langle x, Ry \rangle$ has norm one and satisfies $A(x, y) = 1$. No other real normed linear space has this property ([5], prop. 2.9 and, for the complex case, prop. 2.10).

In general, if x, y belong to the unit sphere of a normed space X, then by applying Theorem 9 to X^* and using the weak* continuity of the 2-homogeneous polynomial xy, it follows that there exists $\varphi \in X^*$ with norm one, such that $\varphi(x)\varphi(y) \geq 1/4$. Hence we have:

PROPOSITION 10. *Let X be a normed space and let x and y be norm-one elements in X. Then there exists a continuous, symmetric bilinear form A on X of norm one such that $A(x, y) \geq 1/4$.*

This can interpreted in terms of symmetric tensor products. Since $X \hat{\bigotimes}_{s,\pi} X$ is the closed subspace of $X \hat{\bigotimes}_\pi X$ generated by the symmetric tensors $x \otimes_s y = (x \otimes y + y \otimes x)/2$, it follows that $\|x \otimes_s y\| \leq \|x\| \, \|y\|$, but, in general, equality does not hold. The proposition implies

$$\tfrac{1}{4} \|x\| \, \|y\| \leq \|x \otimes_s y\| \leq \|x\| \, \|y\|.$$

The same questions for symmetric n-linear forms and n-fold symmetric tensor products can be considered, with similar results.

The closed unit ball of $\hat{X}_\pi^{(n)}$ is the closed, absolutely convex hull of the set $\{x^n : \|x\| = 1\}$, and it was noted earlier that, for real spaces, the closed unit ball is the closed convex hull of $\{\pm x^n : \|x\| = 1\}$. Since the topological nature of the set $\{x^n : x \in X, \ \|x\| = 1\}$ was important in the analysis of extreme points, we now show how the above results can be applied to this set in the case $n = 2$, even if the Banach space is infinite-dimensional. The following result is proved in [12] for spaces with the approximation property.

PROPOSITION 11. *Let X be a Banach space. Then the sets $\{x^2 : x \in X,\ \|x\| = 1\}$ and $\{x^2 : x \in X,\ \|x\| \leq 1\}$ are closed in $\hat{X}_\pi^{(2)}$.*

Proof. We present the proof only for the set $S = \{x^2 : \|x\| = 1\}$, since the remainder of the proof follows from this. Let (x_k^2) be a convergent sequence in S with limit $u \in \hat{X}_\pi^{(2)}$. If the sequence (x_k) in X has a Cauchy subsequence, then it will follow that $u \in S$. Suppose, on the contrary, that (x_k) has no Cauchy subsequence. Then, passing to a subsequence, there exists $\eta > 0$ such that $\|x_k - x_l\| \geq \eta$ for every $k \neq l$. Now, for every $\varepsilon > 0$, there exists K such that $\|x_k^2 - x_l^2\|_\pi < \varepsilon$ for $k, l \geq K$. It follows that

$$\begin{aligned}\|x_k^2 - x_l^2\|_\pi &= \|(x_k - x_l) \otimes_s (x_k + x_l)\| \\ &\geq \frac{1}{4}\|x_k - x_l\|\|x_k + x_l\| > \frac{\eta}{4}\|x_k + x_l\|,\end{aligned}$$

and this implies

$$\|x_k + x_l\| < \frac{\varepsilon}{4\eta} \qquad \text{for every } k, l \geq K, \quad k \neq l.$$

Applying this to the indices k, $k+1$ and $k+1, k+2$ for $k \geq K$ yields $\|x_k - x_{k+2}\| = \|(x_k + x_{k+1}) - (x_{k+1} + x_{k+2})\| < \varepsilon/2\eta$. But we also have $\|x_k + x_{k+2}\| < \varepsilon/4\eta$, and hence $\|2x_k\| = \|(x_k - x_{k+2}) + (x_k + x_{k+2})\| < 3\varepsilon/4\eta$ for every $k \geq K$. But this implies that $x_k \to 0$, which is impossible. This concludes the proof.

(We are grateful to Chris Boyd for drawing our attention to a similar result of Ruess and Stegall ([16], Lemma 1.2).)

This result throws some light on the geometric structure of the closed unit ball of $\hat{X}_\pi^{(2)}$. Recall that a point z_0 in a bounded subset C of a real Banach space Z is *strongly exposed* if there exists $z^* \in Z^*$ such that $z^*(z_0) > z^*(z)$ for every $z \in C \setminus \{z_0\}$ and such that $\lim_n z^*(z_n) = z^*(z_0)$ for a sequence (z_n) in C implies $\lim_n z_n = z_0$. If Z has the Radon–Nikodým property, then every closed bounded convex set in Z is the closed convex hull of its strongly exposed points [9]. If C is the closed convex hull of a *closed* set S, then it is easy to see that all of the strongly exposed points of C lie in S. Hence:

COROLLARY 12. *Let X be a real Banach space. Then the strongly exposed points of the closed unit ball of $\hat{X}_\pi^{(2)}$ are all of the form $\pm x^2$, where $\|x\| = 1$.*

It would be interesting to know when the image of the closed unit ball of X is *weakly* closed in $\hat{X}_\pi^{(n)}$. We can show that this is true, at least when X is a reflexive space with the approximation property.

Let X be a real Banach space. We have seen that the mapping $x \mapsto x^2$ is a continuous 2-homogeneous polynomial from X into $\hat{X}_\pi^{(2)}$. This mapping is $2:1$, except at the origin. We conclude by showing that this is a covering map onto its image:

PROPOSITION 13. *Let X be a real Banach space. The mapping $x \mapsto x^2$ is a local homeomorphism from $X \setminus \{0\}$ onto its image in $\hat{X}_\pi^{(2)}$.*

Proof. Let $x \in X$, $x \neq 0$. Then it is clear that this mapping is injective on the ball $B(x, r)$ if $r < \|x\|$. Fix x and $\varepsilon < \|x\|$. Then $\|x^2 - y^2\| = \|(x + y) \otimes_s (x - y)\| \geq (1/4)\|x + y\|\|x - y\|$. Hence, if $\|x^2 - y^2\| < \varepsilon^2/4$, then $\|x + y\|\|x - y\| < \varepsilon^2$, and so y must lie either in $B(x, \varepsilon)$ or in $B(-x, \varepsilon)$. If we take $y \in B(x, \varepsilon)$, then $\|x + y\| \geq \|2x\| - \|x - y\| > 2\|x\| - \varepsilon$, and so

$$\|x - y\| < \frac{\varepsilon^2}{\|x + y\|} < \frac{\varepsilon^2}{2\|x\| - \varepsilon}.$$

It follows that the restriction of the mapping $x \mapsto x^2$ has a continuous inverse on the ball $B(x, r)$ if $r < \|x\|$.

Obviously, the above results leave many questions only partially answered or unanswered. It is hoped that the results given here will spark an interest in and lead to answers to the problems left open in this paper.

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REFERENCES

1. R. Alencar, R. M. Aron, and S. Dineen, A reflexive space of holomorphic functions in infinitely many variables, *Proc. Amer. Math. Soc.* **90** (1984), 407–411.
2. J. Arias-de-Reyna, “Gaussian Variables, Polynomials and Permanents,” preprint, 1998.
3. R. Aron and S. Dineen, Q-reflexive spaces, *Rocky Mountain J. Math.* (to appear).
4. R. M. Aron, C. Finet, and E. Werner, Some remarks on norm-attaining *n*-linear forms, Function spaces (Edwardsville, IL, 1994), *Lecture Notes in Pure and Appl. Math.* **172** (1995), 19–28.
5. C. Benitez and Y. Sarantopoulos, Characterizations of real inner product spaces by means of symmetric bilinear forms, *J. Math. Anal Appl.* **180** (1993), 207–220.
6. C. Benitez, Y. Sarantopoulos, and A. Tonge, “Lower Bounds for the Norms of Products of Polynomials,” preprint, (1996).
7. M. M. Day, “Normed Linear Spaces,” Springer-Verlag, Berlin/Heidelberg/New York, 1973.

8. J. Diestel, "Geometry of Banach Spaces—Selected Topics," Springer-Verlag, Berlin/Heidelberg/New York, 1975.
9. J. Diestel and J. J. Uhl, Jr., "Vector Measures," Amer. Math. Soc., Providence, RI, 1977.
10. S. Dineen, A Dvoretsky theorem for polynomials, *Proc. Amer. Math. Soc.* **123** (1995), 2817–2821.
11. S. Dineen, "Complex Analysis in Locally Convex Spaces," North-Holland Mathematics Studies, 57. Notas de Matemática, 83. North-Holland, Amsterdam/New York, 1981.
12. M. González and J. Gutiérrez, Gantmacher type theorems for holomorphic mappings, *Math. Nach.* **186** (1997), 131–145.
13. C. Gupta, "Convolution Operators and Holomorphic Mappings on a Banach Space," Séminaire D'Analyse Moderne, No. 2, Dept. Math. Université de Sherbrooke, Quebec, 1969.
14. G. J. O. Jameson, "Summing and Nuclear Norms in Banach Space Theory," Cambridge Univ. Press, Cambridge/New York/New Rochelle/Melbourne/Sydney, 1987.
15. L. Nachbin, "Topology on Spaces of Holomorphic Mappings," Springer-Verlag, New York, 1969.
16. W. M. Ruess and C. P. Stegall, Exposed and denting points in duals of operator spaces, *Israel J. Math.* **53** (1986), 163–190.
17. R. A. Ryan, "Applications of Topological Tensor Products to Infinite Dimensional Holomorphy," Ph.D. Thesis, Trinity College, Dublin, 1980.
18. B. Sims and D. Yost, Linear Hahn–Banach extension operators, *Proc. Edinburgh Math. Soc. (2)* **32** (1989), 53–57.
19. K. Sundaresan, "Geometry of Quadratic Forms," preprint, 1992.